FAMILIES OF HOLOMORPHIC MAPS
INTO RIEMANN SURFACES

BY

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ABSTRACT. In analogy with the Hartogs theorem that separate ana-
lyticity of a function implies analyticity, it is shown that a separately normal
family of holomorphic maps from a polydisk into a Riemann surface is a nor-
mal family. This contrasts with examples of discontinuous separately analytic
maps from a bidisk into the Riemann sphere. The proof uses a theorem on
pseudoconvexity of normality domains, which is proved via the following con-
vergence criterion: a sequence $\{f_j\}$ of holomorphic maps from a complex mani-
fold into a Riemann surface converges to a nonconstant holomorphic map if
and only if the sequence $\{f_j^{-1}\}$ of set-valued maps, defined on the Riemann sur-
face, converges to a suitable set-valued map. Extending Osgood's theorem, it
is also shown that a separately analytic map (resp. a separately normal family
of holomorphic maps) from a polydisk into a hyperbolic complex space is
analytic (resp. normal).

1. A convergence criterion using divisors and level sets. By convention, all
complex spaces (in particular complex manifolds) are connected and second
countable; a Riemann surface is a 1-dimensional complex manifold.

We endow the family $A(M, N)$ of all holomorphic maps from the complex
manifold M into the complex space (usually a Riemann surface) N with the com-
pact-open topology [5, Definition 1.1, p. 257]. Convergence of sequences of
holomorphic maps will always refer to this topology. Now $A(M, N)$ may be re-
garded as a subspace of the space $C(M, N^*)$ of continuous maps from M into the
one-point compactification $N^* = N \cup \{\omega\}$ of N, again endowed with the compact-
open topology. A sequence in $A(M, N)$ is said to *diverge compactly* if it conver-
ges to the map identically ω in $C(M, N^*)$ [18, p. 197]. The topological spaces
N and N^* are metrizable. Equipping N and N^* with metrics, we see that con-
vergence in $A(M, N)$ and $C(M, N^*)$ is equivalent to uniform convergence on com-

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pact sets [5, Theorem 7.2, p. 268]; in turn this is equivalent to continuous convergence [5, 7.5, pp. 268–269].

If g is a holomorphic function defined on a neighborhood of a point a in a complex manifold, we let $v(a, 0, g)$ denote the zero multiplicity (or total order) of g at a [15, p. 156], [7, p. 8]. A function μ, defined on a complex manifold M, is said to be a *nonnegative divisor* if each point a in M possesses an open connected neighborhood U on which there is a holomorphic function g, not identically 0, with $v(\cdot, 0, g) = \mu|U$. The *support* of the nonnegative divisor μ is supp $\mu = \{a \in M|\mu(a) > 0\}$; it is either empty or an analytic set of pure codimension 1 in M. Stoll introduced the concept of convergence of a net of divisors [15, p. 176]. In the special case which we use, his definition reduces to the following. A sequence $\{\mu_j\}$ of nonnegative divisors on the complex manifold M is said to *converge* to the nonnegative divisor μ if for each point a in M there exist an open neighborhood U of a, an integer j_0, and holomorphic functions g_j $(j \geqslant j_0)$ on U satisfying

   (a) $v(\cdot, 0, g_j) = \mu_j|U$ for $j \geqslant j_0$,
   (b) $g_j \longrightarrow g$ $(j \longrightarrow \infty)$,
   (c) $v(\cdot, 0, g) = \mu|U$;

in this case we write $\mu_j \longrightarrow \mu$ $(j \longrightarrow \infty)$.

Hausdorff introduced a topology on the set of all closed subsets of a locally compact Hausdorff space M [9, §28], [15, pp. 196–199]. A point a of M is called a *limit point* of the sequence $\{F_k\}$ of closed subsets if there exist an integer k_0 and points a_k in F_k $(k \geqslant k_0)$ such that $a = \lim a_k$; a point of M is called a *cluster point* of $\{F_k\}$ if it is a limit point of some subsequence of $\{F_k\}$. If the set of limit points coincides with the set of cluster points, $\{F_k\}$ is said to *converge* to this set F, and we write $\lim F_k = F$ or $F_k \longrightarrow F$ $(k \longrightarrow \infty)$.

Let f be a nonconstant holomorphic map from the complex manifold M into the Riemann surface N, and let b be a point of N. Choose a map β taking an open neighborhood V of b in N biholomorphically onto an open subset of $\mathbf{C}$ and satisfying $\beta(b) = 0$. The nonnegative integer

$$v(a, b, f) = v(a, 0, \beta \circ f) \quad \text{if} \quad a \in f^{-1}(V),$$

$$= 0 \qquad\qquad \text{if} \quad a \in f^{-1}(N - V),$$

does not depend on the choice of β; it is called the *b-multiplicity* of f at a. The function $v(\cdot, b, f)$ is a nonnegative divisor on M with support $f^{-1}(b)$.

THEOREM 1. *Let $\{f_j\}$ be a sequence of nonconstant holomorphic maps from the complex manifold M into the Riemann surface N. Then the following are equivalent:*

(1) $\{f_j\}$ converges to a nonconstant map;

(2) there exist nonnegative divisors μ_b $(b \in N)$ on M such that $\{\text{supp } \mu_b\}$ is a partition of M and $v(\cdot, b_j, f_j) \longrightarrow \mu_b$ $(j \longrightarrow \infty)$ for every convergent sequence $b_j \longrightarrow b$ $(j \longrightarrow \infty)$ in N;

(3) M can be partitioned into closed proper subsets M_b $(b \in N)$ such that $f_j^{-1}(b_j) \longrightarrow M_b$ $(j \longrightarrow \infty)$ for every convergent sequence $b_j \longrightarrow b$ $(j \longrightarrow \infty)$ in N.

PROOF. (1) $\Rightarrow$ (2). Assume that $\{f_j\}$ converges to a nonconstant map f. Define $\mu_b = v(\cdot, b, f)$ for $b \in N$. Clearly $\{\text{supp } \mu_b\} = \{f^{-1}(b)\}$ is a partition of M. Let $b_j \longrightarrow b$ $(j \longrightarrow \infty)$ be a convergent sequence in N. Fix a point a of M. In case $f(a) \neq b$, there is an open neighborhood U of a such that $b \notin f(U)$. Shrinking U, we can find an integer j_0 such that $b_j \notin f_j(U)$ for $j \geqslant j_0$. Let g, g_j $(j \geqslant j_0)$ be the constant function identically 1 on U; then all divisors involved are identically 0. In case $f(a) = b$, choose a map β taking an open neighborhood V of a in N biholomorphically onto an open subset of $\mathbf{C}$ and satisfying $\beta(b) = 0$. There exist an index j_0 such that $b_j \in V$ for $j \geqslant j_0$ and a neighborhood U of a such that $f(U) \subset V$. Shrinking U and increasing j_0 if necessary, we may assume that $f_j(U) \subset V$ for $j \geqslant j_0$. Set $g = \beta \circ f|U$, and $g_j = \beta \circ (f_j|U) - \beta(b_j)$ for $j \geqslant j_0$. In either case ($f(a) \neq b$ or $f(a) = b$), we have

$$v(\cdot, 0, g_j) = v(\cdot, b_j, f_j)|U \quad \text{for} \quad j \geqslant j_0,$$

$$g_j \longrightarrow g \; (j \longrightarrow \infty), \quad \text{and} \quad v(\cdot, 0, g) = v(\cdot, b, f)|U = \mu_b|U.$$

Thus $v(\cdot, b_j, f_j) \longrightarrow \mu_b$ $(j \longrightarrow \infty)$.

(2) $\Rightarrow$ (3). Taking $M_b = \text{supp } \mu_b$, this follows from [15, Theorem 4.10, pp. 199–200].

(3) $\Rightarrow$ (1). Assume that (3) holds. If $\{f_j\}$ converges, the limit map must be nonconstant, for otherwise (3) would imply that $M_b = \varnothing$ for all b in N except the value of the limit map. Equip N with a complete metric inducing its topology [5, Corollary 2.4, pp. 294–295]. Suppose $\{f_j\}$ does not converge. Then there is a compact subset K of M such that $\{f_j\}$ does not converge uniformly on K. By the Cauchy criterion, there exist $\epsilon > 0$, subsequences $\{g_k\}$ and $\{h_k\}$ of $\{f_j\}$, and points a_k in K such that $d(g_k(a_k), h_k(a_k)) \geqslant \epsilon$ for all k. By taking a subsequence, we may assume that $a_k \longrightarrow a$ $(k \longrightarrow \infty)$. By (3), $a \in M_b$ for some (uniquely determined) b in N. Let V be a compact neighborhood of b in N having diameter less than ϵ. By taking a subsequence and exchanging the roles of $\{g_k\}$ and $\{h_k\}$ if necessary, we may assume that $h_k(a_k) \notin V$ for all k. By (3), $f_j^{-1}(b) \longrightarrow M_b$ $(j \longrightarrow \infty)$; hence $h_k^{-1}(b) \longrightarrow M_b$ $(j \longrightarrow \infty)$. Since $a \in M_b =$

$\lim h_k^{-1}(b)$, there exist an integer k_0 and points c_k in $h_k^{-1}(b)$ $(k \geq k_0)$ such that $a = \lim c_k$. Choose open connected subsets U_k $(k \geq k_0)$ of M such that $a_k, c_k \in U_k$ and $\{U_k\}$ is a base for the neighborhood system of a. Now $h_k(c_k) = b \in V$ and $h_k(a_k) \notin V$; since h_k is continuous and U_k is connected, there exists a point d_k in U_k such that $h_k(d_k) \in \partial V$. Of course $d_k \longrightarrow a$ $(k \longrightarrow \infty)$; taking a subsequence, we may assume that $e_k = h_k(d_k) \longrightarrow e \in \partial V$ $(k \longrightarrow \infty)$. By (3) again, $h_k^{-1}(e_k) \longrightarrow M_e$ $(k \longrightarrow \infty)$. Thus $a = \lim d_k \in \lim h_k^{-1}(e_k) = M_e$. Since $b \neq e$, we have $M_b \cap M_e = \varnothing$, a contradiction. □

This proof shows how the limit map f determines the nonnegative divisors $\mu_b = \nu(\cdot, b, f)$ and the closed sets $M_b = f^{-1}(b)$. We could have simplified the statement of Theorem 1 by introducing a holomorphic map f and using the divisors and closed sets determined by it. The present statement is easier to use (for example in Theorem 2), since one need not know a priori that the candidate for the limit map is holomorphic. We note that the implication (3) ⇒ (1) works for continuous maps from a locally connected locally compact Hausdorff space into a locally compact metric space. Using similar techniques, we can characterize compact divergence.

THEOREM 1′. *Let $\{f_j\}$ be a sequence of nonconstant holomorphic maps from the complex manifold M into the Riemann surface N. Then the following are equivalent*:

(1) *$\{f_j\}$ diverges compactly*;

(2) *$\nu(\cdot, b_j, f_j) \longrightarrow 0$ $(j \longrightarrow \infty)$ for every convergent sequence $b_j \longrightarrow b$ $(j \longrightarrow \infty)$ in N*;

(3) *$f_j^{-1}(b_j) \longrightarrow \varnothing$ $(j \longrightarrow \infty)$ for every convergent sequence $b_j \longrightarrow b$ $(j \longrightarrow \infty)$ in N*.

2. **Pseudoconvexity of normality domains.** A *Hartogs figure* in $\mathbf{C}^m$ is a pair (H, P) with

$$H = \{w \in \mathbf{C}^m \mid |w_j| < r, \ 1 \leq j \leq m - 1; \ s' < |w_m| < s\}$$
$$\cup \ \{w \in \mathbf{C}^m \mid |w_j| < r', \ 1 \leq j \leq m - 1; \ |w_m| < s\},$$
$$A = \{w \in \mathbf{C}^m \mid |w_j| < r, \ 1 \leq j \leq m - 1; \ |w_m| < s\},$$

where $0 < r' < r$, $0 < s' < s$. An open subset M_0 of the m-dimensional complex manifold M is said to be *pseudoconvex* in M if $\tau(A) \subset M_0$ whenever (H, A) is a Hartogs figure and $\tau: A \longrightarrow \tau(A) \subset M$ is a biholomorphic map satisfying $\tau(H) \subset M_0$. We follow the standard practice of ignoring the biholomorphic map τ and acting as if H and A are subsets of M.

We now recall a few important properties of a Hartogs figure (H, A). By

the Cauchy integral formula, every holomorphic function on H extends (uniquely) to a holomorphic function on A. By extending its reciprocal, we see that the extension of a nowhere zero holomorphic function on H has no zeros in A. It follows that a holomorphic function on H and its extension to A have exactly the same image set. Since a nonempty analytic set of pure codimension 1 in A is the zero set of a holomorphic function on A [7, Lemma 12, p. 251], such a set must intersect H. Finally, if λ and μ are nonnegative divisors on A that agree on H, we can invoke Cousin II [7, Theorem 6, p. 249] to obtain holomorphic functions g and h on A such that $\nu(\cdot, 0, g) = \lambda$ and $\nu(\cdot, 0, h) = \mu$; the meromorphic function g/h is holomorphic and without zeros on H, hence on A, and it follows that $\lambda = \mu$.

A family $\mathcal{F}$ of holomorphic maps from the complex manifold M into the complex space N is called *normal* if every sequence in $\mathcal{F}$ has a subsequence that either converges or diverges compactly [6, Definition 3, p. 114], [18, Definition 1.1, p. 197]. A family $\mathcal{N}$ of nonnegative divisors on the complex manifold M is called *normal* if every net in $\mathcal{N}$ has a convergent subnet [15, p. 154]; in fact $\mathcal{N}$ is normal if and only if every sequence in $\mathcal{N}$ has a convergent subsequence [15, Theorem 2.24, p. 188]. The *normality domain* of the family $\mathcal{F}$ (resp. $\mathcal{N}$) is the union of the open connected subsets U of M such that the restriction $\mathcal{F}|U$ (resp. $\mathcal{N}|U$) is normal. Since normality is a local property, the restriction of a family to any open connected subset of its normality domain is a normal family. Julia [11] proved that the normality domain of a family of holomorphic functions is pseudoconvex, i.e., he proved that the normality domain of $\mathcal{F}$ is pseudoconvex in M in case $N = \mathbf{C}$. In this section we show this holds for any Riemann surface N. The uniformization theorem and [3, Theorem 2, p. 296] give this for any Riemann surface N not biholomorphic to the Riemann sphere, but we shall not need this result. We shall use the fact, proved in [2], that the normality domain of $\mathcal{N}$ is pseudoconvex in M; of course we can interpret this pseudoconvexity result as an extension theorem.

LEMMA 1. *Let (H, A) be a Hartogs figure, and let $\{\mu_j\}$ be a sequence of nonnegative divisors on A. If $\{\mu_j|H\}$ converges (as a sequence of nonnegative divisors on H), then $\{\mu_j\}$ converges.*

PROOF. Let $\mathcal{N} = \{\mu_j | j = 1, 2, 3, \ldots \}$. By hypothesis, H is contained in the normality domain of $\mathcal{N}$. The pseudoconvexity of this normality domain [2, Theorem 12.4, p. 245] shows that $\mathcal{N}$ is a normal family. Thus $\{\mu_j\}$ has a subsequence converging to a nonnegative divisor μ on A. If $\{\mu_j\}$ does not converge to μ, there is a subnet $\{\lambda_t\}$ of $\{\mu_j\}$ such that no subnet of $\{\lambda_t\}$ converges to μ [15, Lemma 2.27, p. 191]. Since $\mathcal{N}$ is a normal family, we may

assume that $\{\lambda_t\}$ converges to a nonnegative divisor λ on A. The divisors λ and μ agree on H, hence $\lambda = \mu$, a contradiction. $\quad\square$

LEMMA 2. *Let $\lambda_j \rightarrow \lambda$ $(j \rightarrow \infty)$ and $\mu_j \rightarrow \mu$ $(j \rightarrow \infty)$ be convergent sequences of nonnegative divisors on the complex manifold M. If (supp λ_j) $\cap$ (supp μ_j) $= \varnothing$ for all j, then (supp λ) $\cap$ (supp μ) is either empty or an analytic set of pure codimension 1 in M.*

PROOF. If the analytic set $X = $ (supp λ) $\cap$ (supp μ) is nonempty, take a point a in X. By the definition of convergence in the space of divisors, there exist an open connected neighborhood U of a, an integer j_0, and holomorphic functions f_j, g_j $(j \geqslant j_0)$ on U such that

$$\nu(\cdot, 0, f_j) = \lambda_j | U, \qquad \nu(\cdot, 0, g_j) = \mu_j | U \quad \text{for} \quad j \geqslant j_0,$$

$$f_j \rightarrow f, \quad g_j \rightarrow g \ (j \rightarrow \infty),$$

$$\nu(\cdot, 0, f) = \lambda | U, \qquad \nu(\cdot, 0, g) = \mu | U.$$

Now $f(a) = g(a) = 0$, $f \not\equiv 0 \not\equiv g$; thus the product fg has total order k at a, where $1 < k < \infty$. By [7, Lemma 2, p. 13] or [17, Lemma 4H, p. 13], there is a coordinate system $(w_1, \cdots, w_{m-1}, z)$ about $a = 0$ such that fg is regular of order k in z. Shrinking U, we may assume that $f(0, z)g(0, z) \neq 0$ if $z \neq 0$ and $(0, z)$ $\in U$.

There exist an open connected neighborhood W of 0 in $\mathbf{C}^{m-1}$ and an open disk Δ around 0 in $\mathbf{C}$ such that $W \times \Delta$ is a relatively compact subset of U and f and g are good for $W \times \Delta$ [17, Definition 5B, pp. 14–15]. Shrinking W and increasing j_0, we may assume that f_j and g_j are good for $W \times \Delta$ if $j \geqslant j_0$. For w in W, let p (resp. q, p_j, q_j) denote the number of zeros of $f(w, \cdot)$ (resp. $g(w, \cdot)$, $f_j(w, \cdot)$, $g_j(w, \cdot)$) in Δ (counting multiplicities); by [17, Lemma 5E, p. 15], these integers do not depend on the choice of w. In view of their integral representation [17, (5.2), p. 15], $p_j \rightarrow p$, $q_j \rightarrow q$ $(j \rightarrow \infty)$. Thus we may assume that $p_j = p$, $q_j = q$ if $j \geqslant j_0$. Using the Weierstrass preparation theorem, we uniquely factor

$$f | W \times \overline{\Delta} = FP, \qquad f_j | W \times \overline{\Delta} = F_j P_j,$$

$$g | W \times \overline{\Delta} = GQ, \qquad g_j | W \times \overline{\Delta} = G_j Q_j,$$

so that P, P_j (resp. Q, Q_j) are Weierstrass polynomials of degree p (resp. q) in z and F, F_j, G, G_j are holomorphic functions without zeros on $W \times \overline{\Delta} (j \geqslant j_0)$. By the Newton identities [16, p. 81], the coefficients in each Weierstrass poly-

nomial are polynomials in the power sums of the roots; in turn, as in [17, pp. 16–17], these power sums can be expressed as integrals involving the factored function. It follows that $P_j \rightarrow P$, $Q_j \rightarrow Q$ $(j \rightarrow \infty)$ in $A(W \times \mathbf{C}, \mathbf{C})$. Let R denote the resultant operator on polynomials of formal degrees p and q [16, pp. 83–84], [17, p. 342]. Define

$$h(w) = R(P(w, \cdot), Q(w, \cdot)),$$

$$h_j(w) = R(P_j(w, \cdot), Q_j(w, \cdot)).$$

Then h, $h_j \in A(W, \mathbf{C})$, and $h_j \rightarrow h$ $(j \rightarrow \infty)$. Since $(\mathrm{supp}\ \lambda_j) \cap (\mathrm{supp}\ \mu_j) = \varnothing$, P_j and Q_j have no common zeros in $W \times \Delta$, hence in $W \times \mathbf{C}$. By the basic property of resultants, the functions h_j have no zeros. On the other hand, $P(0) = Q(0) = 0$, so $h(0) = 0$. By a classical theorem of Hurwitz, $h \equiv 0$. Thus, for every point w in W, $P(w, \cdot)$ and $Q(w, \cdot)$ have a common zero in Δ. Let $\pi\colon \mathbf{C}^{m-1} \times \mathbf{C} \rightarrow \mathbf{C}^{m-1}$ denote the projection. We have just proved that the light holomorphic map $\pi|X \cap (W \times \Delta)\colon X \cap (W \times \Delta) \rightarrow W$ is onto; it follows that this map takes each neighborhood of $a = 0$ in X onto a neighborhood of 0 in W. By [17, Theorem 6B(b), p. 122], $\dim_a X = \dim_0 W = m - 1$. $\square$

THEOREM 2. *Let F be a family of holomorphic maps from the complex manifold M into the Riemann surface N. Then the normality domain of F is pseudoconvex in M.*

PROOF. Let M_0 denote the normality domain of F. Ignoring the implicit biholomorphic map, we take a Hartogs figure (H, A) satisfying $A \subset M$ and $H \subset M_0$; we must prove that $A \subset M_0$. Let $\{h_k\}$ be a sequence in F. Since $H \subset M_0$, there is a subsequence $\{f_j\}$ of $\{h_k\}$ that either diverges compactly or converges on H.

If $\{f_j\}$ diverges compactly on H, implication (1) $\Rightarrow$ (2) of Theorem $1'$ shows that $\nu(\cdot, b_j, f_j)|H \rightarrow 0$ $(j \rightarrow \infty)$ for every convergent sequence $b_j \rightarrow b$ $(j \rightarrow \infty)$ in N. By Lemma 1, we may replace H with A in the last clause. Implication (2) $\Rightarrow$ (1) of Theorem $1'$ completes the proof that $\{f_j\}$ diverges compactly on A.

If $\{f_j|H\}$ converges to a constant map with value b, we will prove that $\{f_j|A\}$ also does. By slimming the Hartogs figure, we may assume that $\{f_j\}$ converges on an open set containing the closure of H. Let V be an open neighborhood of b that is biholomorphically equivalent to an open set in $\mathbf{C}$. Then $V \supset f_j(H) = f_j(A)$ for j sufficiently large. Hence $\{f_j|A\}$ converges to a constant map with value b.

If $\{f_j|H\}$ converges to a nonconstant map g, we may assume that f_j is nonconstant for all j. Let $b_j \rightarrow b$ $(j \rightarrow \infty)$ be a convergent sequence in N. By

implication (1) $\Rightarrow$ (2) of Theorem 1, $\nu(\cdot, b_j, f_j)|H \longrightarrow \nu(\cdot, b, g)$ $(j \longrightarrow \infty)$; by
Lemma 1, there is a nonnegative divisor μ_b on A such that $\nu(\cdot, b_j, f_j)|A \longrightarrow \mu_b$
$(j \longrightarrow \infty)$. Since $\mu_b|H = \nu(\cdot, b, g)$, the divisor μ_b depends only on the point b
and not on the particular sequence $b_j \longrightarrow b$ $(j \longrightarrow \infty)$ used in its definition. We
now show that the family $\{\text{supp } \mu_b\}$ is a partition of A. If b and c are distinct
points of N,

$$(\text{supp } \nu(\cdot, b, f_j)|A) \cap (\text{supp } \nu(\cdot, c, f_j)|A)$$

$$= A \cap f_j^{-1}(b) \cap f_j^{-1}(c) = \varnothing \quad \text{for all } j.$$

By Lemma 2, the analytic set $X = (\text{supp } \mu_b) \cap (\text{supp } \mu_c)$ is either empty or has
pure codimension 1 in M. But $X \cap H = g^{-1}(b) \cap g^{-1}(c) = \varnothing$, hence $X = \varnothing$,
proving that $\{\text{supp } \mu_b\}$ is disjoint. To see that this family covers A, we fix a
point a in A and define $d_j = f_j(a)$. Then $a \in f_j^{-1}(d_j) = \text{supp } \nu(\cdot, d_j, f_j)$. Since
the pure 1-codimensional analytic set $f_j^{-1}(d_j)$ contains the point a of A, it inter-
sects H; the convergence of $\{f_j\}$ on H then implies the convergence of a subse-
quence of $\{d_j\}$ to a point d in N. Using Lemma 1 and implication (1) $\Rightarrow$ (2)
of Theorem 1, we see that the corresponding subsequence of $\{\nu(\cdot, d_j, f_j)|A\}$
converges to a nonnegative divisor μ on A. The divisors μ and μ_d agree on H,
hence on A. Thus $a \in \text{supp } \mu = \text{supp } \mu_d$. By implication (2) $\Rightarrow$ (1) of The-
orem 1, $\{f_j|A\}$ converges. $\square$

3. **A Hartogs theorem for normal families of maps.** A classical theorem
of Hartogs asserts that a separately holomorphic function is holomorphic [8],
[10, Theorem 2.2.8, pp. 28–29]. Let D denote the open unit disk $\{z \in \mathbf{C}|$
$|z| < 1\}$. A *coordinate disk* in the polydisk D^m is a submanifold of the form
$\{(d_1, \cdots, d_{k-1})\} \times D \times \{(d_{k+1}, \cdots, d_m)\}$ where $d_j \in D$. A map from a
polydisk D^m into a complex space is said to be *separately holomorphic* if its
restriction to each coordinate disk is holomorphic. In precise terms, the Hartogs
theorem says that if $N = \mathbf{C}$, every separately holomorphic map $f: D^m \longrightarrow N$ is
holomorphic. In view of the loose statement that "separate analyticity implies
analyticity," it is natural to expect that the conclusion of the Hartogs theorem
holds for an arbitrary complex space (or at least complex manifold) N. However,
such a theorem fails if $N = \mathbf{C} \cup \{\infty\}$ is the Riemann sphere: the map $f: D^2 \longrightarrow$
$\mathbf{C} \cup \{\infty\}$ defined by

$$f(w, z) = (w + z)^2/(w - z) \quad \text{if} \quad w \neq z,$$

$$= \infty \quad \text{if} \quad w = z \neq 0,$$

$$= 0 \quad \text{if} \quad w = z = 0,$$

is separately holomorphic but is not continuous at $(0, 0)$. Of course the problem lies in continuity. If f is known to be continuous, N may be replaced by a domain in $\mathbf{C}^n$, and separate analyticity implies analyticity by Osgood's lemma [7, Theorem 2, pp. 2–3]. We shall use the following generalized version of the Hartogs theorem.

LEMMA 3. *Let N be a complex manifold whose universal covering manifold is Stein. Then every separately holomorphic map f: $D^m \longrightarrow N$ is holomorphic.*

PROOF. Proceeding by induction on m, we may assume that every separately holomorphic map from D^{m-1} into N is holomorphic. Since the conclusion is local, it is enough to prove that f is holomorphic in a neighborhood of 0. Let U be a neighborhood of $f(0)$ in N that is biholomorphically equivalent to an open set in $\mathbf{C}^n$, and let V be an open neighborhood of $f(0)$ whose closure is contained in U. Regard D^m as $D^{m-1} \times D$. By hypothesis, $f(0, \cdot)$: $D \longrightarrow N$ is continuous; hence there is an open disk Δ centered at 0 such that $f(\{0\} \times \Delta) \subset V$. Take z in Δ; by the induction assumption, $f(\cdot, z)$: $D^{m-1} \longrightarrow N$ is continuous; thus there exists $r(z) > 0$ such that $f(w, z) \in V$ if $|w_j| < r(z)$ $(j = 1, \cdots, m - 1)$. This means that the union of the sets $B(k) = \{z \in \Delta | f(w, z) \in V$ if $|w_j| < 1/k$ $(j = 1, \cdots, m - 1)\}$ $(k = 1, 2, \cdots)$ is Δ. By the Baire category theorem, there is an integer k_0 such that the closure of $B(k_0)$ contains a nonempty, open, connected, simply connected set Z. Let $W = \{w \in \mathbf{C}^{m-1} | |w_j| < 1/k_0\}$. If $(w, z) \in W \times Z$, z is a limit of points z' in $B(k_0)$ satisfying $f(w, z') \in V$; the continuity of $f(w, \cdot)$ shows that $f(w, z) \in \overline{V} \subset U$. By Osgood's theorem [10, Lemma 2.2.9, p. 28], [14], $f|W \times Z$: $W \times Z \longrightarrow U \subset N$ is holomorphic. Let p: $L \longrightarrow N$ be the universal covering manifold of N. Since $W \times Z$ is simply connected, there exists a holomorphic map g: $W \times Z \longrightarrow L$ such that $p \circ g = f|W \times Z$. Choose a point z_0 in Z, and fix it for the rest of this proof. For each w in W, the holomorphic map $f(w, \cdot)$: $D \longrightarrow N$ can be lifted to a holomorphic map $h(w, \cdot)$: $D \longrightarrow L$ with $p \circ h(w, \cdot) = f(w, \cdot)$ and $h(w, z_0) = g(w, z_0)$; by uniqueness of lifting, $h(w, z) = g(w, z)$ if $z \in Z$. Using the imbedding theorem for Stein manifolds [7, Theorem 13, p. 226], we may regard L as a complex submanifold of a euclidean space $\mathbf{C}^q$. Consider the map

$$h: W \times D \longrightarrow L \subset \mathbf{C}^q;$$

$h|W \times Z = g|W \times Z$ is holomorphic and $h(w, \cdot)$: $D \longrightarrow L$ is holomorphic for each w in W. Applying the Hartogs lemma [10, Lemma 2.2.11, pp. 28–29] to its coordinates, we see that h is holomorphic. Thus $p \circ h = f|W \times D$ is holomorphic. $\square$

A family of holomorphic maps from a polydisk into a complex space is

said to be *separately normal* if its restriction to each coordinate disk is normal. Nishino has proved a normal families version of the Hartogs theorem: a separately normal family of holomorphic functions is normal [13, Théorème II, pp. 263–264]. In view of the failure of the Hartogs theorem for a single map into the Riemann sphere, it is somewhat surprising that Nishino's theorem extends to Riemann surfaces.

THEOREM 3. *If $\mathcal{F}$ is a separately normal family of holomorphic maps from a polydisk D^m into a Riemann surface N, then $\mathcal{F}$ is a normal family.*

PROOF. By the uniformization theorem, the universal covering surface of N is the Riemann sphere, $\mathbf{C}$, or D.

Assume that the universal covering surface of N is the Riemann sphere. Then up to biholomorphic equivalence, $N = \mathbf{C} \cup \{\infty\}$. Let $\{f_j\}$ be a sequence in $\mathcal{F}$. If $f_j \equiv \infty$ for infinitely many j, we can extract a subsequence converging to the constant map identically ∞. Otherwise, there is a subsequence $\{g_k\}$ of $\{f_j\}$ such that $g_k \not\equiv \infty$ for all k. Now $G = \{g_k | k = 1, 2, 3 \cdot \cdot \cdot\}$ may be regarded as a separately normal family of meromorphic functions on D^m; by [13, Théorème IV, pp. 279–282], this family is normal as a family of meromorphic functions. Extracting a subsequence, we may assume that $\{g_k\}$ converges as a sequence of meromorphic functions on D^m, i.e., [13, p. 273] there is an analytic set S of codimension at least 2 in D^m such that $\{g_k | D^m - S\}$ converges in $A(D^m - S, N)$. Thus $D^m - S$ is contained in the normality domain of G; by Theorem 2, this normality domain must equal D^m. It follows that a subsequence of $\{f_j\}$ converges.

Assume that the universal covering surface of N is $\mathbf{C}$ or D. Let $\{f_j\}$ be a sequence in $\mathcal{F}$. There exists a subsequence $\{g_k\}$ of $\{f_j\}$ such that $\{g_k(0)\}$ converges to a point of N (or to ω in the one-point compactification of N). If infinitely many of the maps g_k are constant, it is easy to extract a convergent (resp. compactly divergent) subsequence; thus we may assume that g_k is nonconstant for all k. Write $D^m = D^{m-1} \times D$. Extracting a subsequence, we may assume that $\{g_k(0, \cdot)\}$ converges (resp. diverges compactly) in $A(D, N)$. Let E be a countable dense subset of D. Proceeding by induction on m, we may assume that the restriction of $\mathcal{F}$ to each polydisk $D^{m-1} \times \{z\}$ ($z \in D$) is a normal family. Extracting successive subsequences and taking a diagonal, we may assume that $\{g_k(\cdot, z)\}$ converges (resp. diverges compactly) in $A(D^{m-1}, N)$ for every z in E. Take w in D^{m-1}; since $\{g_k(w, \cdot)\}$ consists of maps in a normal family and converges pointwise in N (resp. to ω) on the dense subset E of D, this sequence converges (resp. diverges compactly) in $A(D, N)$. Similarly, $\{g_k(\cdot, z)\}$ converges (resp. diverges compactly) in $A(D^{m-1}, N)$ for each z in D. Define

$g: D^m \longrightarrow N \cup \{\omega\}$ by $g(w, z) = \lim g_k(w, z)$. Either $g(D^m) \subset N$ or $g \equiv \omega$; in the first case we conclude from Lemma 3 that g is holomorphic. Suppose no subsequence of $\{g_k\}$ converges to g in $C(D^m, N^*)$. Then there exists a convergent sequence $a_k \longrightarrow a$ ($k \longrightarrow \infty$) in D^m such that $g_k(a_k) \nrightarrow g(a)$ ($k \longrightarrow \infty$). Extracting a subsequence we may assume that $b_k = g_k(a_k) \longrightarrow b$ ($k \longrightarrow \infty$), where $g(a) \neq b \in N^*$. There is a convex open neighborhood G of a in D^m such that $b \notin g(G)$. (If $g \equiv \omega$ or any other constant, we may even take $G = D^m$.) Let $V_k = g_k^{-1}(b_k) \cap G$. The sequence $\{V_k\}$ consists of pure $(m - 1)$-dimensional analytic sets in G. If H is a coordinate disk in D^m, $g_k|H \cap G \longrightarrow g|H \cap G$ ($k \longrightarrow \infty$) in $C(H \cap G, N^*)$; hence $\{V_k \cap H\}$ is a locally finite sequence. By [1, Theorem 4, pp. 339–340], $\{V_k\}$ is locally finite. But for k sufficiently large, $a_k \in V_k$ and $a_k \longrightarrow a$ ($k \longrightarrow \infty$), a contradiction. $\square$

4. **Maps into higher dimensional complex manifolds.** In [3, Theorem 2, p. 296] it was shown that the normality domain of a family of holomorphic maps from an m-dimensional complex manifold M into an n-dimensional complex manifold N is $(m - n)$-pseudoconvex in M, provided the universal covering manifold of N is Stein. In view of Theorem 2, it seems that the condition on the universal covering manifold may be superfluous. Unfortunately, Theorem 1 does not have an obvious generalization to maps into higher dimensional complex manifolds, so an attempt at a proof along the same lines seems futile.

The statement of Theorem 3 becomes false if the words "Riemann surface" are replaced by "complex manifold." Take $F = \{f_j | j = 1, 2, \cdots\}$ where $f_j: D^2 \longrightarrow \mathbb{C}^2$ is defined by $f_j(w, z) = j^2(w - j^{-1}, z - j^{-1})$. The restriction of $\{f_j\}$ to each coordinate disk diverges compactly, hence F is separately normal. But $f_j(j^{-1}, j^{-1}) = (0, 0)$, so F cannot be a normal family. The difficulty here is caused by the relative dimensions. To handle this, we define an n-dimensional *coordinate polydisk* in D^m ($1 \leqslant n \leqslant m$) to be the cartesian product of n copies of D and $m - n$ singleton subsets of D. The following theorem can be proved using the technique in the last paragraph of the proof of Theorem 3. Again, it seems that the assumption about the universal covering manifold may be unnecessary.

THEOREM 4. *Let F be a family of holomorphic maps from a polydisk D^m into a complex manifold of dimension $n \leqslant m$ whose universal covering manifold is Stein. If the restriction of F to each n-dimensional coordinate polydisk in D^m is a normal family, then F is a normal family.*

Finally, we note that strong Hartogs-type theorems hold for maps into (Kobayashi) hyperbolic complex spaces [12, pp. 97–98]. Since these spaces may be regarded as generalizations of the unit disk, the following might more

properly be called an Osgood-type theorem [14].

THEOREM 5. *Let N be a hyperbolic complex space. Then*

(1) *every separately holomorphic map f: $D^m \to N$ is holomorphic;*

(2) *every separately normal family $\bar{F}$ of holomorphic maps from D^m into N is normal.*

PROOF. (1) By Osgood's lemma [7, Theorem 2, pp. 2–3], it will suffice to prove that f is continuous. Using the triangle inequality and the fact that holomorphic maps from D into N are distance decreasing with respect to the Poincaré distance ρ on D and the Kobayashi distance d_N on N, we obtain the inequality $d_N(f(z), f(a)) \leqslant \Sigma \rho(z_j, a_j)$ for all z and a in D^m. Thus f is continuous as a map of D^m into the metric space (N, d_N). Since d_N induces the standard topology on N [4], f is continuous.

(2) Let $\{f_j\}$ be a sequence in $\bar{F}$. As in the beginning of the last paragraph of the proof of Theorem 3, we can find a subsequence $\{g_k\}$ of $\{f_j\}$ such that $g(z) = \lim g_k(z) \in N^*$ exists for all z in D^m. Either $g(D^m) \subset N$ or $g \equiv \omega$; in the first case we use (1) to conclude that g is holomorphic. The inequality $d_N(g_k(z), g_k(a)) \leqslant \Sigma \rho(z_j, a_j)$ and pointwise convergence of $\{g_k(a)\}$ to $g(a)$ now imply the continuous convergence of $\{g_k\}$ to g. $\square$

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